where the first integral on the right-hand side can now be identified with the gamma function[4] and the second one with the incomplete gamma function, so that†

$$q = (CV_E^{M-1}k^{-N}/\beta \sin\gamma_E)[\Gamma(N) - \Gamma(N,k)] \quad (7)$$

Ratios of the incomplete to the complete gamma function are shown in Fig. 1 for several values of N; it can be seen that this ratio becomes small for $k \dot{>} 4$. Physically, this corresponds to bodies which have decelerated to a small fraction of their entry velocity at impact. Thus, for $V_F \ll V_E$

$$q \doteq (CV_E^{M-1}k^{-N})/(\beta \sin\gamma_E)\Gamma(N) \quad (8a)$$

or, in terms of the parameters of primary interest

$$q \sim (m/C_DA)^N(\beta \sin\gamma_E)^{N-1} \quad (8b)$$

where only the exponent of the density, N, appears. When the body's final velocity is not small

$$q \sim (m/C_DA)^N(\beta \sin\gamma_E)^{N-1}[\Gamma(N) - \Gamma(N,k)] \quad (8c)$$

For the limiting case of a nearly constant velocity entry‡ ($V \doteq V_E$), it can easily be shown that for an exponential atmosphere

$$dt \doteq -dy/V_E \sin\gamma_E = d\bar{\rho}/\bar{\rho}\beta V_E \sin\gamma_E \quad (9)$$

Substituting Eqs. (4) and (9) into Eq. (5) and integrating yields

$$q \doteq CV_E^{M-1}/N\beta \sin\gamma_E \quad (10a)$$

or

$$q \dot{\sim} (\beta \sin\gamma_E)^{-1} \quad (10b)$$

The altitude and velocity of maximum heating can also readily be found by substituting Eq. (3) into (4), setting the derivative of Eq. (4), with respect to $\bar{\rho}$ equal to zero and solving, to yield

$$\bar{\rho} = 2N/MB \quad (11a)$$

and

$$V/V_E = e^{-N/M} \quad (11b)$$

The following conclusions are noted.

1) The total heat input is substantially reduced by decreasing m/C_DA, except in the limiting case of bodies which do not decelerate significantly during entry, when heating becomes independent of m/C_DA.

2) For a point on the body experiencing only laminar flow, such as the stagnation point, the total heating decreases significantly with increasing entry angle. (However, increasing the entry angle also increases the Re, since $Re \sim (m/C_DA)(\beta \sin\gamma_E)$ so that keeping γ_E and, therefore Re, small enough to preserve laminar flow, generally results in considerably less heat input, for bodies with $V_F \ll V_E$, than using steep entry angles if turbulent boundary-layer flow occurs at high speed.)[3] For laminar flow, the maximum heating rate occurs at $V/V_E \doteq 0.85$.

3) For areas on a body experiencing predominantly turbulent flow during entry, the total heat input still decreases with increasing entry angle, but much more slowly than for laminar flow. If transition to turbulence occurs early in the entry (roughly true if $B < 20$), then peak heating occurs at about $V/V_E \doteq 0.80$. However, for larger values of B, transition and, consequently, peak turbulent heating will occur nearer the speed for maximum Reynolds number which is $V/V_E \doteq 0.37$ or considerably later during the entry.

4) For bodies experiencing predominantly radiative heating, it appears that shallow entries always reduce the total heat input. Although the value of N can vary considerably with flight conditions, atmospheric composition, etc., it is probably always greater than one, for cases of practical interest. Peak radiative heating in air occurs roughly at $V/V_E \doteq 0.89$

† Because constant values of N and M were used during the integration leading to Eq. (7), only continuum flow heating processes have been considered. This is approximately valid for $B < 10^3/\sin\gamma_E$ and in general excludes only very small bodies, roughly of centimeter size or less.

‡ For instance, for a steep ($\sin\gamma_E \doteq 1$) Earth atmospheric entry such that $V_F = 0.9V_E$, a spherical iron body of 4m radius, or 5° half-angle cone 2.5m long, having a specific gravity of one, is required.

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Optimization of Search for an Object Drifting in Outer Space

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A SPACECRAFT that becomes unable to use its own power in outer space starts drifting and becomes a subject of search. We shall assume that the search for the drifting object is carried out by using equipment with limited range of detection. If the location and the velocity of the object at the moment when the drifting starts were known to the searchers, the future track of the object would be calculated accurately and the object would be located by using this information. In the present paper we shall consider the problem involved in optimizing the search for a drifting object whose location and speed at the moment when the drifting starts are known only approximately. Furthermore, we shall assume that the search may be expressed in terms of the so-called search density. An expression for the optimal search density will be derived.

Basic Equations

We shall assume that the location $\mathbf{x}(0)$ and the velocity $\mathbf{v}(0)$ of an object in outer space, at time 0, possess the probability densities $g[\mathbf{x}(0)]$ and $h[\mathbf{v}(0)]$, respectively, and that $\mathbf{x}(t)$, the location of the object at time t, satisfies a differential equation

$$\ddot{\mathbf{x}} = \mathbf{f}(\dot{\mathbf{x}},\mathbf{x}) \quad (1)$$

where the dot denotes the time derivative, which is solvable in a closed form or numerically. One now wishes to organize

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the search for the object so that the probability of detection becomes as close to one as possible. We shall consider the case where the searching effort may be represented through a search density $\varphi(\mathbf{x},t)$ (cf. Refs. 1 and 2) with the following properties: 1) $\varphi(\mathbf{x},t) \geqslant 0$, 2) $\int \varphi(\mathbf{x},t)dV_x = \Phi =$ const for all $t > 0$ and 3) $\varphi(\mathbf{x},t)\Delta t + o(\Delta t)$ is the probability that the object will be found during time $(t, t + \Delta t)$, given that the object is at point $\mathbf{x}$.

By solving Eq. (1) one obtains relations

$$\mathbf{x} = \mathbf{F}[t; \mathbf{x}(0),\mathbf{v}(0)] \tag{2}$$

and

$$\mathbf{v} = \mathbf{V}[\mathbf{x}(t),\mathbf{x}(0),\mathbf{v}(0)] \tag{3}$$

From (2) one obtains

$$\mathbf{x}(0) = \mathbf{a}[\mathbf{x}(t),\mathbf{v}(0)] \tag{4}$$

Let us denote the tangent vector and the normal vector of track $\mathbf{x}(t)$ through $\mathbf{v}_0(t)$ and $\mathbf{n}_0(t)$, respectively, $[|\mathbf{v}_0(t)| = |\mathbf{n}_0(t)| = 1$ and $\mathbf{v}_0(t)\cdot\mathbf{n}_0(t) = 0]$. Let the volume

$$V[t,\mathbf{x}(t)] = \pi[\delta(t)\mathbf{n}_0(t)]^2|l(t)\mathbf{v}_0(t)| \tag{5}$$

be assumed to move with the object along trajectory $\mathbf{x}(t)$ so that $\mathbf{x}(0)\epsilon V[0,\mathbf{x}(0)]$ implies $\mathbf{x}(t)\epsilon V[t,\mathbf{x}(t)]$ and vice versa. Then it is easy to show that

$$\dot{V}[t,\mathbf{x}(t)] = V[t,\mathbf{x}(t)][\mathbf{v}_0(t); \mathbf{v}_0(t) + 2\mathbf{n}_0(t); \mathbf{n}_0(t)]: [\nabla; \mathbf{v}(t)] \tag{6}$$

where [cf. Eq. (3)] $\nabla;\ \mathbf{v}(t) = \nabla_\mathbf{x};\ \mathbf{v}[\mathbf{x}, \mathbf{x}(0), \mathbf{v}(0)]$ [here the notation of dyadic calculus was used. For instance: let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ be vectors, then $\mathbf{a}\cdot(\mathbf{b};\mathbf{c}) = (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$ and $(\mathbf{a};\mathbf{b}):(\mathbf{c};\mathbf{d}) = (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$].

We now proceed to derive $u[\mathbf{x},t;\ \mathbf{v}(0)]$ the probability density of the object at time t, given that its velocity at time 0 was $\mathbf{v}(0)$. Obviously with $V\epsilon[t,\mathbf{x}(t)] = \pi\{[\epsilon\delta(t)\mathbf{n}_0(t)]\}^2 \times |\epsilon l(t)\mathbf{v}_0(t)| = \epsilon^3 V[t,\mathbf{x}(t)]$ [cf. Eq. (5)]

$$\int_{V\epsilon[t,x(t)]} u[\mathbf{x},t;\ \mathbf{v}(0)]dV_x = \int_{V\epsilon[0,\mathbf{x}(0)]} g[\mathbf{x}(0)]dV_{x(0)} \tag{7}$$

where $g(\mathbf{x})$ is the probability density of the object at time 0. For small ϵ Eq. (7) now implies that

$$u[\mathbf{x},t;\ \mathbf{v}(0)] = g[\mathbf{x}(0)]\{V[0,\mathbf{x}(0)]/V[t,\mathbf{x}(t)]\} = g\{\mathbf{a}[\mathbf{x}(t),\ \mathbf{v}(0)]\}(V\{0,\mathbf{a}[x(t),\ \mathbf{v}(0)]\})/V[t,\mathbf{x}(t)] \tag{8}$$

where Eq. (4) was applied and where $V[t,\mathbf{x}(t)]$ satisfies Eq. (6). Function (8) is now the probability density of the location of the object, given that no search has taken place. One can think here of a "probability cloud" that is moving in the space with density given through $u[\mathbf{x},t;\ \mathbf{v}(0)]$. We shall use the term probability cloud in the following considerations. We shall now derive an equation for $u[\mathbf{x},t;\ \mathbf{v}(0),\varphi]$, the probability density of the location of the target at time t, given that the initial velocity was $\mathbf{v}(0)$ and that the search during time $(0,t)$ with search density $\varphi(\mathbf{x},t)$ has not been successful. Because according to Eq. (8) $u(\mathbf{x},t;\ \mathbf{v}(0))V(t,\mathbf{x}) =$ const, it follows through differentiation that $\dot{u}[\mathbf{x},t;\ \mathbf{v}(0)] = -u[\mathbf{x},t;\ \mathbf{v}(0)]\ [\dot{V}(\mathbf{x},t)/V(\mathbf{x},t)]$ which yields, since obviously $\dot{u}[\mathbf{x},t;\mathbf{v}(0)] = (\partial/\partial t)u[\mathbf{x},t;\ \mathbf{v}(0)] + \mathbf{v}\cdot\nabla u[\mathbf{x},t;\ \mathbf{v}(0)]$ the equation

$$(\partial/\partial t)u[\mathbf{x},t;\ \mathbf{v}(0)] = -\mathbf{v}\cdot\nabla u[\mathbf{x},t;\ \mathbf{v}(0)] - [\ln V(t,\mathbf{x})]^{\cdot}u[\mathbf{x},t;\ \mathbf{v}(0)] \tag{9}$$

The contribution of an unsuccessful search during time $(t, t + \Delta t)$ to the probability density of the target can be derived heuristically as follows (for a rigorous treatment see Ref. 3). Assume that the target is stationary during time $(t, t + \Delta t)$ with probability density $u(\mathbf{x},t)$ and that search during this interval of time by applying search density $\varphi(\mathbf{x},t)$ has not yielded results. Then the probability density of the target at time $t + \Delta t$ is given through Bayes' formula:

$$u(\mathbf{x},t + \Delta t) = u(\mathbf{x},t)[1 - \varphi(\mathbf{x},t)\Delta t][1 - \Delta t \int_R u(\xi,t)\varphi(\xi,t)dV_\xi]^{-1} + o(\Delta t)$$

i.e., the change of $u(x,t)$ at point $\mathbf{x}$, due to searching during time $(t, t + \Delta t)$, is

$$u(\mathbf{x},t + \Delta t) - u(\mathbf{x},t) = u(\mathbf{x},t) \times [\textstyle\int_R u(\xi,t)\varphi(\xi,t)dV_\xi - \varphi(\mathbf{x},t)]\Delta t + o(\Delta t) \tag{10}$$

On the other hand, in the absence of search, Eq. (9) implies that $u[\mathbf{x},t,\mathbf{v}(0)]$ changes at point $\mathbf{x}$, because of the motion of the probability cloud, as follows:

$$u[\mathbf{x},t + \Delta t,\mathbf{v}(0)] - u[\mathbf{x},t,\mathbf{v}(0)] = (-\mathbf{v}\cdot\nabla u[\mathbf{x},t;\ \mathbf{v}(0)] - \{\ln V[t,\mathbf{x}(t)]\}^{\cdot}u[\mathbf{x},t;\ \mathbf{v}(0)])\Delta t + o(\Delta t) \tag{10a}$$

Equations (10) and (10a) now suggest that the effect of search on the probability density is taken into account by adding the term $u[\mathbf{x},t;\ \mathbf{v}(0),\varphi]\{\int_R u[\xi,t;\ \mathbf{v}(0),\varphi]\varphi(\xi,t)dV_\xi - \varphi(\mathbf{x},t)\}$ to the right-hand side of Eq. (9)

$$(\partial/\partial t)u[\mathbf{x},t;\ \mathbf{v}(0),\varphi] = -\mathbf{v}\cdot\nabla u[\mathbf{x},t;\ \mathbf{v}(0),\varphi] - u[\mathbf{x},t;\ \mathbf{v}(0),\varphi]\dot{V}(t,\mathbf{x})/V(t,\mathbf{x}) + u[\mathbf{x},t;\ \mathbf{v}(0),\varphi] \times [\textstyle\int_R u[\xi,t;\ \mathbf{v}(0),\varphi]\varphi(\xi,t)dV_\xi - \varphi(\mathbf{x},t)] \tag{11}$$

The probability that the object will be located during time $(0,t)$ given that initial velocity was $\mathbf{v}(0)$ and that search density $\varphi(\mathbf{x},t)$ was applied, becomes (see Ref. 2)

$$P[\mathbf{v}(0)] = 1 - \exp\left\{-\int_0^t d\tau \int_R dV_\xi \varphi(\xi,\tau) \times u[\xi,\tau;\mathbf{v}(0),\varphi]\right\} \tag{12}$$

In the same way as in Ref. 2 we now substitute

$$u[\mathbf{x},t;\mathbf{v}(0),\varphi] = y[\mathbf{x},t;\mathbf{v}(0),\varphi] \times \exp\left\{-\int_0^t d\tau \int_R \varphi(\xi,\tau)u[\xi,\tau;\ \mathbf{v}(0),\varphi]dV_\xi\right\} \tag{13}$$

Equations (11) and (12) become then

$$(\partial/\partial t)y[\mathbf{x},t;\ \mathbf{v}(0),\varphi] = -\mathbf{v}\cdot\nabla y[\mathbf{x},t;\ \mathbf{v}(0),\varphi] - \{[\ln V(t,\mathbf{x})]^{\cdot} + \varphi(\mathbf{x},t)\}y[\mathbf{x},t;\ \mathbf{v}(0),\varphi] \tag{14}$$

and

$$P[\mathbf{v}(0)] = 1 - \textstyle\int_R y[\mathbf{x},t;\ \mathbf{v}(0),\varphi]dV_x \tag{15}$$

respectively. Let $W(t,\mathbf{x})$ be defined by equation

$$[\ln W(t,\mathbf{x})]^{\cdot} = [\ln V(t,\mathbf{x})]^{\cdot} + \varphi(\mathbf{x},t) \tag{16}$$

i.e., let

$$W(t,\mathbf{x}) = V(t,\mathbf{x}) \exp\left[\int_0^t \varphi(\mathbf{x},\tau)d\tau\right] \tag{17}$$

Then a comparison of Eqs. (10) and (14) shows that

$$W(t,\mathbf{x})y[\mathbf{x},t;\ \mathbf{v}(0),\varphi] = \text{const} = g[\mathbf{x}(0)]V[0,\mathbf{x}(0)] \tag{18}$$

It now follows immediately that

$$y[\mathbf{x},t;\ \mathbf{v}(0),\varphi] = u[\mathbf{x},t;\ \mathbf{v}(0)] \exp\left[-\int_0^t \varphi(\mathbf{x},\tau)d\tau\right] \tag{19}$$

where $u[\mathbf{x},t;\mathbf{v}(0)]$, given by Eq. (8), is the probability density of the location of the object given that the initial velocity was $\mathbf{v}(0)$ and that no search has taken place. We now obtain from Eqs. (15) and (19) by unconditioning with respect to $\mathbf{v}(0)$ the following expression for the probability that the target will be located during time $(0,t)$

$$P = 1 - \int_R h(\mathbf{v})dV_v \int_R u(\mathbf{x},t;\mathbf{v}) \exp\left[-\int_0^t \varphi(\mathbf{x},\tau)d\tau\right] = 1 - \int_R u(\mathbf{x},t) \exp\left[-\int_0^t \varphi(x,\tau)d\tau\right]dV_x \tag{20}$$

where

$$u(\mathbf{x},t) = \int_R h(\mathbf{v})u(\mathbf{x},t;\mathbf{v})dV_v \quad (21)$$

with $h(\mathbf{v})$ the probability density of $\mathbf{v}$.

Optimal Search Density

We now have the following problem of optimization; to be found is function $\varphi(\mathbf{x},\tau)$, $\tau\epsilon[0,t]$, such that

$$\int_R u(\mathbf{x},t) \exp\left(-\int_0^t \varphi(\mathbf{x},\tau)d\tau\right)dV_x = \min \quad (22)$$

By writing

$$\psi(\mathbf{x}) = \int_0^t \varphi(\mathbf{x},\tau)d\tau \quad (23)$$

we arrive at the familiar problem of optimization[1]; to be found is function $\psi(\mathbf{x})$ such that

$$\int_R u(\mathbf{x},t) \exp[-\psi(\mathbf{x})]dV_x = \min \quad (24)$$

while

$$\int_R \psi(\mathbf{x})dV_x = \Phi t \quad (25)$$

and

$$\psi(\mathbf{x}) \geqslant 0 \quad (26)$$

Here t is to be considered as a constant. Problem (24–26) is readily solved[1]

$$\psi(\mathbf{x}) = \int_0^t \varphi(\mathbf{x},\tau)d\tau = \theta[u(\mathbf{x},t) - \lambda] \times \ln[u(\mathbf{x},t)/\lambda] \quad (27)\dagger$$

where λ must be chosen such that

$$\Phi t = \int_R \theta[u(\mathbf{x},t) - \lambda] \ln[u(\mathbf{x},t)/\lambda] \quad (28)$$

This yields the following differential equation for the determination of λ

$$(\dot{\lambda}/\lambda)A(\lambda,t) - \int_{A(\lambda,t)} [\ln u(\mathbf{x},t)]^{\cdot}dV_x + \Phi = 0 \quad (29)$$

Here $A(\lambda,t)$ denotes the volume of set $[\mathbf{x}:u(\mathbf{x},t) > \lambda]$. $A(\lambda,0)$ is most likely given in actual cases.

With λ obtained from Eq. (29), Eq. (27) yields the following expression for the search density

$$\begin{aligned}\varphi(\mathbf{x},t) &= \theta[u(\mathbf{x},t) - \lambda] \ln[u(\mathbf{x},t)/\lambda]^{\cdot} \\ &= \theta[u(\mathbf{x},t) - \lambda]\{[\ln u(\mathbf{x},t)]^{\cdot} - \dot{\lambda}/\lambda\} \quad (30)\end{aligned}$$

Clearly the optimal search density $\varphi(\mathbf{x},t)$ is moving along with the probability cloud. It is to be remembered that in Eqs. (29) and (30) $[\ln u(\mathbf{x},t)]^{\cdot} = (\partial/\partial t) \ln u(\mathbf{x},t) + \mathbf{v}\cdot\nabla \ln u(\mathbf{x},t)$. Equations related to Eqs. (29) and (30) were obtained by Arkin[4] in the case of a stationary target.

An Illustration

The purpose of the following example is to illustrate the nature of the theory just given. In order to get expressions in a closed form, far-going simplifications will be made.

We shall consider the motion of an object of mass m in the field of gravitation of a large spherical mass M of radius R. Let us place the origin of an (x,y,z) coordinate system at the center of the sphere. Let us assume that the object is moving along a line that passes through the center of the mass M and that the location of the object, at time 0, is given through the probability density

$$u(\mathbf{x},0) = (2\pi)^{-3/2}\sigma_1^{-3} \exp(-\tfrac{1}{2}\{[x - x_c(0)]^2 + y^2 + z^2\}\sigma_1^{-2}) \quad (31)$$

while $\mathbf{v}(0)$, the speed of the object at time 0, is known exactly. Now

$$m\ddot{x}(t) = -kMmx^{-2}(t) \quad (32)$$

i.e.,

$$\tfrac{1}{2}v^2(t) - kM/x(t) = \tfrac{1}{2}v^2(0) - kM/x(0) \quad (33)$$

Equation (6) becomes now, as is easily seen,

$$\dot{V}[t,x(t)] = V[t,x(t)][dv(x)/dx] \quad (34)$$

We shall make the following simplifying assumptions: 1) $\sigma_1 \ll R$, i.e., the particles of the probability cloud [Eq. (31)] move practically along paths parallel to the x axis and 2) $(kM/x) + \frac{1}{2}v^2(0) - kM/x(0) \approx kM/x$ for all $x > x(0)$, i.e., the total mechanical energy of the particle is very small.

It now follows from Eq. (33) that

$$(d/dt)x(t) = \alpha[1/x(t) + \beta]^{1/2} \approx \alpha[1/x(t)]^{1/2} \quad (35)$$

where we wrote more briefly $\alpha = (2kM)^{1/2}$ and $\beta = (\frac{1}{2}kM)[v(0) - 2kM/x(0)]$. The approximate Eq. (35) gives

$$x(t) = [3\alpha t/2 + x^{3/2}(0)]^{2/3} \quad (36)$$

In particular $x_c(t) = x_c(0)[1 + (\alpha,t)]^{2/3}$ where we wrote more briefly $\alpha_1 = 3\alpha/[2x_c^{3/2}(0)]$. It follows now easily from Eqs. (8, 31, and 34) that

$$u(x,t) = (2\pi)^{-3/2}\sigma_1^{-2}\sigma_{2t}^{-1} \exp\left[-\tfrac{1}{2}\left(\{[x - x_c(t)]^2/\sigma_{2t}^2\} + [(y^2 + z^2)/\sigma_1^2]\right)\right] \quad (37)$$

where $\sigma_{2t} = \sigma_1[1 + (\alpha,t)]^{-1/3}$. It is seen that the originally spherically symmetric probability density becomes flat in the direction of motion as the probability cloud moves on.

We now proceed to determine the optimal search density $\varphi(\mathbf{x},t)$. As is immediately clear from Eqs. (37) and (27), the volume $A(\lambda,t)$ is the volume of the ellipsoid

$$[x - x_c(t)]^2/\sigma_{2t1}^2 + y^2/\sigma_{11}^2 + z^2/\sigma_{11}^2 = 1 \quad (38)$$

where

$$\sigma_{11} = \sigma_1\rho_0 \quad (39)$$

and

$$\sigma_{2t1} = \sigma_{2t}\rho_0 \quad (40)$$

with

$$\rho_0 = \{-2 \ln[(2\pi)^{3/2}\sigma_1^2\sigma_{2t}\lambda]\}^{1/2} \quad (41)$$

It is seen from Eqs. (39–41) that

$$A(\lambda,t) = (\tfrac{4}{3})\pi\sigma_1^2\sigma_{2t}\rho_0^3 \quad (42)$$

Equation (27) becomes now

$$\int_0^t \varphi(\mathbf{x},\tau)d\tau = \theta[\rho_0^2 - (\xi^2 + \mu^2 + \eta^2)] \times \tfrac{1}{2}[\rho_0^2 - (\xi^2 + \mu^2 + \eta^2)] \quad (43)$$

where we introduced the dimensionless quantities $\xi = [x - x_c(t)]/\sigma_{2t}$, $\mu = y/\sigma_1$ and $\eta = z/\sigma_1$. Condition (28) yields the equation

$$\Phi t = \tfrac{1}{5}A(\lambda,t)\rho_0^2 \quad (44)$$

which gives, with help of Eq. (42), the following expression for ρ_0: $\rho_0 = [(15\Phi t)/(4\pi\sigma_1^2\sigma_{2t})]^{1/5}$. Finally we obtain from Eq. (43) that

$$\begin{aligned}\varphi(\mathbf{x},t) &= (d/dt)\{\theta[\rho_0^2 - (\xi^2 + \mu^2 + \eta^2)] \times \tfrac{1}{2}[\rho_0^2 - (\xi^2 + \eta^2 + \mu^2)]\} \quad (45) \\ &= \theta[\rho_0^2 - (\xi^2 + \mu^2 + \eta^2)]\rho_0\dot{\rho}_0\end{aligned}$$

As a check we calculate $\int_R \varphi(\mathbf{x},t)dV_x$

$$\int_R \varphi(\mathbf{x},t)dV_x = \int_R (d/dt)\{\theta[\rho_0^2 - (\xi^2 + \mu^2 + \eta^2)] \times \tfrac{1}{2}[\rho_0^2 - (\xi^2 + \mu^2 + \eta^2)]\}dV_x = (d/dt) \iiint \sigma_1^2\sigma_{2t}\theta\,[\rho_0^2 - (\xi^2 + \mu^2 + \eta^2)]\tfrac{1}{2}[\rho_0^2 - (\xi^2 + \mu^2 + \eta^2)]d\xi d\eta d\mu = (d/dt)[\tfrac{1}{5}A(\lambda,t)\rho_0^2] = \Phi$$

† $\theta(X) = 1$ for $X \geqslant 0$; 0 for $X < 0$.

It is seen from Eq. (45) that the optimal search density is moving with the probability cloud. At time t the search density is equal to $\rho_0\hat{\rho}_0$ throughout ellipsoid

$$[x - x_c(t)]^2/\sigma_{2t}^2 + y^2/\sigma_1^2 + z^2/\sigma_1^2 = \rho_0^2$$

where $x_c(t) = x_c(0)(1 + \alpha_1 t)^{2/3}$ and $\sigma_{2t} = \sigma_1(1 + \alpha_1 t)^{-1/3}$. Our result is related to one of the examples by Arkin,[4] although in our case the target is moving.

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RFI Measurements on a LES-7 Prototype Pulsed Plasma Thruster

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In recent years electric propulsion technology has been successfully used in space application. Pulsed plasma thrusters (PPT) were flown on the LES-6 satellite to provide thrust for station-keeping and station-changing functions. A thruster with ten times as much power is being developed for LES-7. These units are manufactured by the Republic Division of Fairchild-Hiller Corporation for Lincoln Laboratory. Possible interference with the communication system is a matter of concern. The operation of PPT's is inherently "noisy" and radio frequency interference (RFI) effects on the communication system are now well-known. Although observations indicate that RFI did not significantly affect system performance of LES-6, there is greater concern over RFI emitted from the more powerful units. This Note gives the result of RFI measurements made at X-band on a LES-7 prototype thruster.

Measurement of RFI

The LES-7 prototype used in these measurements was developed for the Wright-Patterson Air Force Base, Aero Propulsion Laboratory. The unit was measured as received without a metal case to suppress RFI. It was mounted in a 4-ft-diam vacuum chamber that was lined with broadband absorber material (Fig. 1). Thus, the measurements are relatively free from chamber resonances or other extraneous signals.

The PPT was operated under normal operating conditions firing once very 1.5 sec and was measured as though it were a transmitter emitting a pulse of noise at some time set by the trigger.

Basically the thruster consists of a 20-joule storage capacitor shunted across the upper and lower electrodes (the cathode and anode) of the output nozzle. Behind this nozzle is a teflon fuel bar 5 in. long and 1 in. × 1 in. in cross section. Just in front of each fuel bar and located in the upper electrode is a spark plug which is used to initiate the high-voltage discharge of the capacitor. Within 3 μsec the main capacitor discharges all of its energy, ionizing teflon particles and accelerating the ionized particles through the nozzle to provide thrust. After recharge, a pulse reinitiates the process and it is repeated at a regular pulse frequency.

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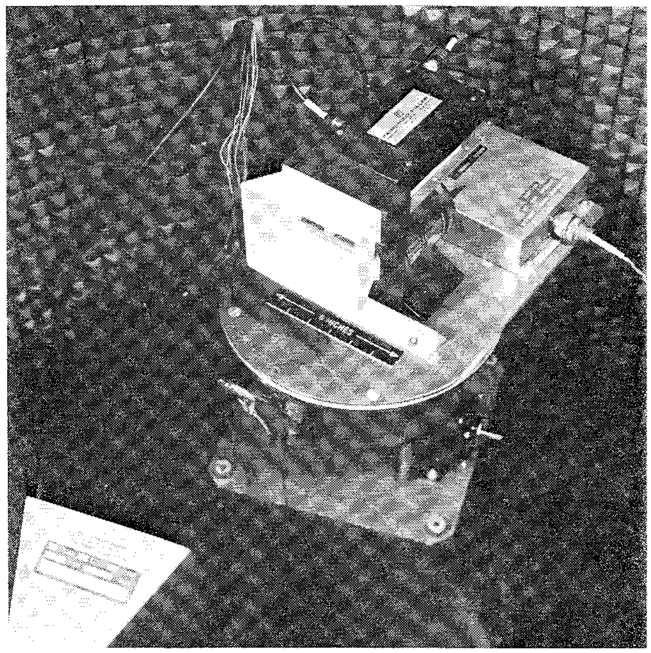

Fig. 1 Thruster in chamber.

Initially, a low noise receiver was used with a Hewlett-Packard spectrum analyzer. These measurements, at UHF and at X band, did not provide reliable data for a number of reasons. There was no way to synchronize the PPT firing with a spectral display, and the noise was so broad that components could be isolated almost anywhere from 100 MHz to 10 GHz. Attempts to measure the individual pulses constituting the RFI also failed, because of instrument limitations. However, it appeared that most of the RFI was associated with the main capacitor discharge, because both occurred 2.0 μsec after the triggering process; on the other hand, it was not contained in the ensuing plasma cloud, because its duration was ~1 μsec, whereas the luminescence of the plasma cloud persists for 15 to 20 μsec. The level of the RFI changes 5 to 10 db with successive discharges, in random fashion; this phenomenon may or may not indicate that the thrust of the PPT varies with successive discharges. The measurement block diagram that was finally used to measure the characteristics of the RFI is shown in Fig. 2.

Figure 2 shows the X-band horn, a 100-MHz bandwidth filter centered at 8 GHz and an X-band detector with 100-MHz video bandwidth. The scope display was triggered by the PPT trigger pulse. The noise pulses were recorded to be of 1.0-μsec duration with a peak power of +4.5 dbm at the PPT. The noise from the thruster is directional; it emanates from the throat in a cardioidal pattern in the horizontal plane.

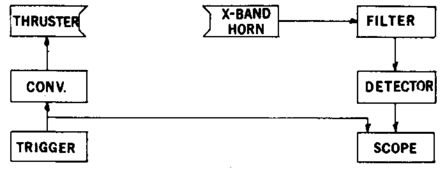

Fig. 2 Block diagram.